NOTES ON THE CYCLIC QUADRILATERAL.

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A triangle gives rise to certain well-defined and unique points, such as the centroid, symmedian point, etc., and these points may be considered as attached to the triangle. Four vertices will give rise to four such triangles, successively obtained by omitting each vertex in turn, and there is a natural inquiry into the behavior of the four sets of attached points. In general the relations are not always simple; but there are certain pretty simplifications when the four vertices are all on a circle.

If we call four points on a circle α, β, γ, δ, the four triangles will be $\alpha\beta\gamma$, $\beta\gamma\delta$, $\gamma\delta\alpha$, $\delta\alpha\beta$. By definition they all have the same circumcenter, o. This suggests a treatment by vector analysis, in which we take o as origin and consider the circle as a unit circle or base circle. Then α, β, γ, δ are orthogonal numbers, or *turns*, upon this circle.

It may be convenient to introduce the symmetric functions of α, β, γ, δ. For three points α, β, γ these are

$$s_1 = \alpha + \beta + \gamma, \qquad s_2 = \alpha\beta + \beta\gamma + \gamma\alpha, \qquad s_3 = \alpha\beta\gamma.$$

Similarly for four points the symmetric functions are the sums taken one, two, three, and four at a time. The context makes it clear as to whether the symmetric functions are for three or four points.

Besides the single points attached to the triangle, of which we have cited the centroid and symmedian point as examples, there are certain pairs of points. The most interesting of these are the Hessian points and the equiangular or Fermat points. These will give rise to simple configurations when considered for the four triangles which make up an inscribed quadrilateral. And finally the particular set of four points formed by the incenters of a triangle gives rise to a recently studied rectangular net when considered for the four triangles of an inscribed quadrilateral.

Although all of the proofs might be thrown into the notation of vector analysis, it will be found more convenient in some cases to indicate other methods. It may be said that the facts were largely suggested by the writer, while the methods of proof were generally intimated by his father, Professor Morley.

1. **Centroids.** The centroid of the triangle α, β, γ may be called g_δ,

and is by definition

$$x = \frac{\alpha + \beta + \gamma}{3}.$$

This may be written as

$$3x = \alpha + \beta + \gamma + \delta - t$$

or

$$3x = s_1 - t,$$

where t is a variable turn traveling round the base circle. The equation is now *symmetrized.* For a varying t it represents a circle, and when t picks up the particular values α, β, γ, δ, x is in succession the point g_α, g_β, g_γ, g_δ. Hence *the four centroids are on a circle,* with center $s_1/3$ and radius $1/3$.

Moreover,

$$3g_\alpha = s_1 - \alpha, \qquad 3g_\beta = s_1 - \beta,$$

so that

$$3(g_\alpha - g_\beta) = -(\alpha - \beta).$$

From this we see that $g_\alpha g_\beta$ is parallel to $\alpha\beta$ and one-third of its length. Hence *the four centroids form a cyclic quadrilateral similar to the four vertices and parallel in situation, but of one-third the size.*

2. **Orthocenters.** The orthocenter of a triangle, h_δ, is given by

$$x = \alpha + \beta + \gamma.$$

This may be symmetrized and written

$$x = s_1 - t.$$

This is again a circle, and when t picks up the value α, β, γ, δ, x is in succession h_α, h_β, h_γ, h_δ. Hence *the four orthocenters are on a circle,* with center s_1 and radius 1. Moreover,

$$h_\alpha - h_\beta = -(\alpha - \beta),$$

so that *the four orthocenters form a cyclic quadrilateral equal to the quadrilateral formed by the four vertices, and parallel in situation.*

3. **Centers of nine-point circles.** The center of the nine-point circle, n_δ, is found to be

$$x = \frac{\alpha + \beta + \gamma}{2}.$$

This may be symmetrized and written for the four triangles as

$$2x = s_1 - t.$$

It follows that *the centers of the four nine-point circles form a cyclic quadrilateral similar to the quadrilateral formed by the four vertices, and parallel in situation, but of half the size.*

4. **Symmedian points.** The symmedian point, k, of a triangle (here the symmetric functions are for three variables) is found by direct calculation to be

$$x = \frac{6s_1s_3 - 2s_2}{9s_3 - s_1s_2}.$$

It is not convenient to symmetrize this expression, and our previous treatment breaks down.

It is then advisable to find out how the symmedian point appears in barycentric coördinates. Since we are to deal with a quadrilateral, symmetry will be gained by choosing the diagonal triangle for reference. Then the four vertices will have coördinates

$$\begin{array}{llll} \alpha: & a_0 & a_1 & a_2 \\ \beta: & -a_0 & a_1 & a_2 \\ \gamma: & a_0 & -a_1 & a_2 \\ \delta: & a_0 & a_1 & -a_2. \end{array}$$

These four points are to lie on the circle apolar to the reference triangle,

$$c_0x_0^2 + c_1x_1^2 + c_2x_2^2 = 0, \tag{1}$$

where the c's are the cotangents of the angles of the reference triangle.

Let us calculate the symmedian point of $\beta\gamma\delta$. The tangent to (1) at β is

$$-c_0a_0x_0 + c_1a_1x_1 + c_2a_2x_2 = 0,$$

and at γ is

$$c_0a_0x_0 - c_1a_1x_1 + c_2a_2x_2 = 0.$$

Any line through their intersection is

$$-c_0a_0x_0 + c_1a_1x_1 + c_2a_2x_2 + \lambda c_0a_0x_0 - \lambda c_1a_1x_1 + \lambda c_2a_2x_2 = 0.$$

This line passes through δ if λ is so chosen that

$$-c_0a_0^2 + c_1a_1^2 - c_2a_2^2 + \lambda c_0a_0^2 - \lambda c_1a_1^2 - \lambda c_2a_2^2 = 0.$$

By virtue of (1) this reduces to

$$c_1a_1^2 + \lambda c_0a_0^2 = 0.$$

Substituting this value of λ and dividing by $c_0a_0^2c_1a_1^2c_2a_2^2$, we have for the symmedian line through δ the equation

$$\frac{-c_0a_0x_0 + c_1a_1x_1 + c_2a_2x_2}{c_1a_1^2c_2a_2^2} = \frac{c_0a_0x_0 - c_1a_1x_1 + c_2a_2x_2}{c_2a_2^2c_0a_0^2} = m,$$

where m is unaltered by interchange of letters, as appears immediately

when the letters are permuted cyclically to form the equations of the symmedian lines through β and γ. By adding numerators and denominators in the equation written,

$$\frac{2c_2a_2x_2}{c_2a_2^2(c_0a_0^2 + c_1a_1^2)} = m$$

and again using the reduction (1),

$$x = \frac{-m}{2} c_2a_2^3.$$

The symmedian point is then found to have as coördinates simply

$$k_0 = c_0a_0^3, \qquad k_1 = c_1a_1^3, \qquad k_2 = c_2a_2^3.$$

The four symmedian points derived from the four triangles are therefore

$$\begin{array}{lrrr} k: & k_0 & k_1 & k_2 \\ k_\beta: & -k_0 & k_1 & k_2 \\ k_\gamma: & k_0 & -k_1 & k_2 \\ k_\delta: & k_0 & k_1 & -k_2. \end{array}$$

Comparison with the coördinates of α, β, γ, δ shows that *the four symmedian points have the same diagonal triangle as the four vertices.*

This may also be seen from the theorem that *there is a particular projection which sends a circle with an inscribed quadrilateral into a circle with an inscribed rectangle.* This is proved by Professor Morley as follows: Let v be the exterior diagonal of the four points α, β, γ, δ on a circle C in a plane P. Take a sphere on α, β, γ, δ. Draw either tangent plane from v to the sphere, and let N be the point of contact. Take any plane P' parallel to this tangent plane. When we project from N the circle C will become a circle C' in P', and also the four points α, β, γ, δ will become the vertices of a parallelogram, since the third or exterior diagonal has gone to infinity. Thus the projection of the circle with the inscribed quadrilateral α, β, γ, δ is a circle with an inscribed parallelogram; i.e., a rectangle.

When the circle C is projected into a circle C', the tangents of C are projected into the tangents of C', and therefore the symmedian point of a triangle inscribed in C goes into the symmedian point of the triangle in C'. When we project four points on C into a rectangle on C', the four symmedian points become the four symmedian points of the triangles formed from the rectangle. But these form a concentric rectangle, or a rectangle having the same diagonal triangle. Hence, by projecting back, the original symmedian points have the same diagonal triangle as the four concyclic vertices.

Returning to the barycentric expressions, we derived a simple cubic transformation which sends the inscribed quadrilateral into its symmedian points, namely

$$k_i = c_i a_i^3.$$

In this the a's may be eliminated, with the resulting locus for k,

$$c_0^{1/3} x_0^{2/3} + c_1^{1/3} x_1^{2/3} + c_2^{1/3} x_2^{2/3} = 0. \tag{2}$$

The relation of this locus and the circle

$$c_0 x_0^2 + c_1 x_1^2 + c_2 x_2^2 = 0, \tag{1}$$

savors strongly of the familiar case in rectangular coördinates of the astroid

$$X^{2/3} + Y^{2/3} = A^{2/3}, \tag{3}$$

and its circumcircle

$$X^2 + Y^2 = A^2. \tag{4}$$

In fact the symmedian locus (2) is the particular projection of the astroid in which the circle (4) goes into the circle (1).

It is then clear that the four symmedian points of an inscribed quadrilateral are on a six-cusped curve whose cusps are on the sides of the diagonal triangle. This curve, together with the diagonal triangle, affords a unique construction for the symmedian quadrilateral when one symmedian point is given.

There is another way in which the peculiarity of the symmedian quadrilateral may be stated. Any four points a_i set up a pencil of conics. The polar of a point x with respect to the pencil is a pencil of lines through y. Hence x and y are in a quadratic Cremona involution

$$x_i y_i = a_i^2.$$

This is a transformation over the plane. In particular, it sends the orthocenter of the diagonal triangle, namely

$$y_0 = c_1 c_2, \qquad y_1 = c_2 c_0, \qquad y_2 = c_0 c_1, \tag{5}$$

into

$$x_0 = c_0 a_0^2, \qquad x_1 = c_1 a_1^2, \qquad x_2 = c_2 a_2^2. \tag{6}$$

When the four points a_i are on the circle (2), (6) is at infinity, and the transformation sends the orthocenter to infinity. But when the four points setting up the involution are k_i, (6) becomes

$$x_0 = (c_0 a_0^2)^3, \qquad x_1 = (c_1 a_1^2)^3, \qquad x_2 = (c_2 a_2^2)^3;$$

a point on the cubic

$$x_0^{1/3} + x_1^{1/3} + x_2^{1/3} = 0. \tag{7}$$

Hence the symmedian points are such that their involution sends the orthocenter of their diagonal triangle into a point on the cubic (7).

5. **Hessian points.** A Hessian point may be defined as a point whose images in the sides of a triangle form an equilateral triangle. Reverting to vector analysis, this will lead to the expression*

$$h_\delta = -\frac{\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta}{\alpha + \omega^2\beta + \omega\gamma}.$$

There will be two such points for any triangle; the other point of the pair, h_δ', is found by interchanging ω and ω^2, where ω and ω^2 are the complex cube roots of 1.

The above expression is for the triangle formed by omitting δ. The corresponding Hessian, h_α, for the triangle formed by omitting α will be found by cyclically permuting the letters,

$$h_\alpha = -\frac{\gamma\delta + \omega^2\delta\beta + \omega\beta\gamma}{\beta + \omega^2\gamma + \omega\delta}.$$

But if we multiply this second expression by ω^2 above and below, it will be seen to be the same as the first, except that α is replaced by δ. Hence $h_\delta - h_\alpha$ will have a factor $(\delta - \alpha)$, and may be written

$$h_\delta - h_\alpha = \frac{(\delta - \alpha)[\beta\gamma(1 - \omega + \omega^2)\cdot + \omega(\beta^2 + \gamma^2)]}{(\alpha + \omega^2\beta + \omega\gamma)(\beta + \omega^2\gamma + \omega\delta)}$$

$$= \frac{\omega(\delta - \alpha)(\beta - \gamma)^2}{(\alpha + \omega^2\beta + \omega\gamma)(\beta + \omega^2\gamma + \omega\delta)}.$$

In a similar way

$$h_\beta - h_\gamma = \frac{\omega(\beta - \gamma)(\delta - \alpha)^2}{(\gamma + \omega^2\delta + \omega\alpha)(\delta + \omega^2\alpha + \omega\beta)},$$

and similarly for the other differences. The double ratio for the Hessians h_α, h_β, h_γ, h_δ, is then

$$\frac{(h_\delta - h_\alpha)(h_\beta - h_\gamma)}{(h_\alpha - h_\beta)(h_\gamma - h_\delta)} = \frac{(\delta - \alpha)^3(\beta - \gamma)^3}{(\alpha - \beta)^3(\gamma - \delta)^3},$$

and the double ratio for the four points α, β, γ, δ is

$$\frac{(\delta - \alpha)(\beta - \gamma)}{(\alpha - \beta)(\gamma - \delta)}.$$

Since α, β, γ, δ are on a circle, this last double ratio is real; hence the double ratio for the Hessians, being the cube of this, is also real. A similar result will be obtained for the set of points h_α', h_β', h_γ', h_δ'. Therefore *each set of Hessian points is on a circle.*

* Harkness and Morley, Theory of Functions, p. 26.

6. **Equiangular points.** An old problem, attributed to Fermat, is to find a point the sum of whose distances from three vertices, say α, β, γ, is a minimum. At such a point the angles subtended by $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ will be equal or supplementary. There are in fact two points, familiar as the intersections of the lines joining α, β, γ to the vertices of equilateral triangles described, all outwards or all inwards, on $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$. Another definition would show the points to be the isogonal conjugates of the Hessian pair. They are variously known as Fermat points, isogonal centers,* or equiangular points.

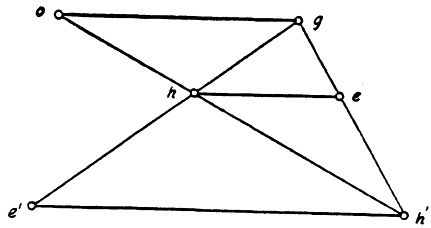

Fig. 1.

To handle the equiangular points by vector analysis, a convenient starting point is Figure 1, taken from a paper by Professor Morley.† Here o is the circumcenter of any triangle, g the centroid, h, h' the Hessian pair, and e, e' the equiangular points. Immediately it is seen that

$$e - g = (h' - g)\frac{h}{h'}.$$

Substituting the previous values for g, h, and h', we have

$$3e_\delta = \alpha + \beta + \gamma - (\alpha + \omega^2\beta + \omega\gamma)\frac{\beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta}{\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta}.$$

Let us write

$$3(e_\delta - \alpha) = \frac{(\beta+\gamma-2\alpha)(\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta)-(\alpha+\omega^2\beta+\omega\gamma)(\beta\gamma+\omega^2\gamma\alpha+\omega\alpha\beta)}{\beta\gamma+\omega\gamma\alpha+\omega^2\alpha\beta}.$$

If in this expression α were equal to β, the numerator would vanish, which is convenient to note geometrically; so that $(\alpha - \beta)$ is a factor. Similarly $(\alpha - \gamma)$ is a factor, and the expression may be written

$$3(e_\delta - \alpha) = \frac{\omega(\omega - \omega^2)(\omega\gamma - \beta)(\alpha - \beta)(\alpha - \gamma)}{\beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta}.$$

We may now permute the letters and write (since $\omega - \omega^2 = i\sqrt{3}$),

$$i\sqrt{3}(e_\delta - \beta) = \frac{\omega(\gamma - \omega\alpha)(\beta - \gamma)(\beta - \alpha)}{\gamma\alpha + \omega\alpha\beta + \omega^2\beta\gamma},$$

and also

$$i\sqrt{3}(e_\alpha - \beta) = \frac{\omega(\gamma - \omega\delta)(\beta - \gamma)(\beta - \delta)}{\gamma\delta + \omega\delta\beta + \omega^2\beta\gamma}.$$

* Neuberg, Sur les projections . . . d'un triangle fixe, Académie de Belgique, t. XLIV.
† Quarterly Journal, vol. 25 (1891), p. 186.

The difference will reduce to

$$i\sqrt{3}(e_\delta - e_\alpha) = \frac{\omega(\beta-\gamma)(\delta-\alpha)[A]}{B_\delta \cdot B_\alpha},$$

where

$$A = [(\delta\alpha + \beta\gamma)(\omega\gamma + \omega^2\beta) + \beta\gamma(\delta + \alpha)],$$

and

$$B_\delta = \gamma\alpha + \omega\alpha\beta + \omega^2\beta\gamma.$$

Similarly

$$i\sqrt{3}(e_\beta - e_\gamma) = \frac{\omega(\beta-\gamma)(\delta-\alpha)[A']}{B_\beta \cdot B_\gamma},$$

where

$$A' = [(\beta\gamma + \delta\alpha)(\omega^2\delta + \omega\alpha) + \delta\alpha(\beta + \gamma)].$$

The double ratio of the four equiangular points

$$\frac{(e_\delta - e_\alpha)(e_\beta - e_\gamma)}{(e_\alpha - e_\beta)(e_\gamma - e_\delta)},$$

will then have the form

$$\frac{(\beta-\gamma)^2(\delta-\alpha)^2[AA']}{B_\alpha \cdot B_\beta \cdot B_\gamma \cdot B_\delta} \times \frac{B_\alpha \cdot B_\beta \cdot B_\gamma \cdot B_\delta}{(\alpha-\beta)^2(\gamma-\delta)^2[CC']},$$

where C and C' are expressions similar in formation to A and A'. The factor

$$\frac{(\beta-\gamma)^2(\delta-\alpha)^2}{(\alpha-\beta)^2(\gamma-\delta)^2}$$

is real, since α, β, γ, δ are on a circle; moreover, since the conjugate of A is $\bar{A}$, the conjugate of the expression AA'/CC' is

$$\frac{\bar{A}\bar{A}'}{\bar{C}\bar{C}'} = \frac{\alpha^2\beta^2\gamma^2\delta^2 AA'}{\alpha^2\beta^2\gamma^2\delta^2 CC'};$$

thus the double ratio of the equiangular points is equal to its conjugate, and hence real. This will be true for the points e_i' as well as for e_i; therefore *each set of equiangular points is on a circle.*

7. **Incenters.** The triangle $\beta\gamma\delta$ will have four incenters, using the term in the general sense, of type I_α. The fact that the sixteen points I_α, I_β, I_γ, I_δ form a rectangular net has been discussed so recently, makes a bare reference sufficient.*

* The theorem was cited by Neuberg, in 1906; more recent proofs have been given by F. V. Morley, Amer. Math. Monthly, vol. 24, p. 430 (1917); N. Altshiller, Am. Math. Monthly, vol. 25, p. 412 (1918); and in the comprehensive article by J. W. Clawson, Annals of Math., vol. 20, p. 254 (1919). See also F. V. Morley, Amer. Math. Monthly, June 1920.